Introduction to Higher Topoi

4: Topologies and geometric morphisms

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The story so far:

- ∞-topoi
- descent
- local classes + universal families
- truncation and connectivity

n-groupoids :

$\mathcal{S}_{\leq n} \subseteq \mathcal{S}$ full subcategory of n-truncated objects

"n-groupoids"

$\mathcal{S}_{\leq 0} = \mathrm{Set}$ $\quad$ $\mathcal{S}_{\leq -1} = \mathrm{Prop} = \{0 < 1\}$

$\mathcal{S}_{\leq n}$ is an

(n+1)-category ($\simeq$ (n+1,1)-category) :

$\mathcal{C}$ is an (n+1)-category iff all $\mathrm{Map}_{\mathcal{C}}(x,y) \in \mathcal{S}_{\leq n}$

Presheaves of (n-1)-groupoids: $\mathcal{C}$ ∞-category (small)

$$\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}_{\leq n-1}) \simeq \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})_{\leq n-1} = \mathrm{PSh}(\mathcal{C})_{\leq n-1}$$

$$\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}_{\leq n-1}) \simeq \mathrm{Fun}((h_n\mathcal{C})^{op}, \mathcal{S}_{\leq n-1})$$

<u>**Note:**</u> Every ∞-category admits $\mathcal{C} \xrightarrow{\tau_n} h_n\mathcal{C}$ (n-category) initial among functors to n-categories

n-topos : ∞-category $\mathcal{E}$ such that $\exists$

(1) small $\cancel{\infty}$ n-category C, and

(2) accessible fully faithful $i : \mathcal{E} \hookrightarrow \underline{PSh(C)_{\leq n-1}}$, with

(3) left adjoint $\ell \dashv i$ such that

(4) ℓ preserves finite limits

Examples : 1-topos = Grothendieck topos $\qquad$ 0-topos = (frame, c.H.a) locale

Example: $\mathcal{E}$ ∞-topos $\Longrightarrow$ $\mathcal{E}_{\leq n-1}$ n-topos

Proof: Pick

$$\begin{tikzcd}
\mathcal{E} \arrow[r, hook, "i"'] & \mathrm{PSh}(C) \arrow[l, bend right, "\ell \ \text{lex}"'] \\
\mathcal{E}_{\leq n-1} \arrow[u, hook] \arrow[r, hook, "i"'] & \mathrm{PSh}(C)_{\leq n-1} \arrow[u, hook] \arrow[l, bend right, "\ell"']
\end{tikzcd}$$

$f \in \mathrm{Trunc}_n \Leftrightarrow \Delta^{n+2}(f) \in \mathrm{Iso}$

ℓ, i preserve this cond.

$\Longrightarrow$ $\mathcal{E}_{\leq 0}$ a Grot. topos

$\mathcal{E}_{\leq -1} = \mathrm{Sub}(1)$ locale.

Fact: Every n-topos is $\simeq \mathcal{E}_{\le n-1}$ for some ∞-topos $\mathcal{E}$

Warning: $\mathcal{E}_{\le 0} \simeq \mathcal{F}_{\le 0} \not\Longrightarrow \mathcal{E} \simeq \mathcal{F}$

$$X \in \mathcal{S} \quad \rightsquigarrow \quad \left(\mathcal{S}_{/X}\right)_{\le 0} \simeq \mathrm{Fun}\big(\underbrace{(\Pi_1 X)}_{\text{fund groupoid}}{}^{op}, \mathrm{Set}\big)$$

Note:

$$\begin{tikzcd}
\mathcal{E}_{\le -1} \arrow[r, hook] & \mathcal{E}_{\le 0} \arrow[l, bend right] \arrow[r, hook] & \mathcal{E}_{\le 1} \arrow[l, bend right, "\mathrm{Tr}_0"'] \arrow[r, hook] & \mathcal{E}_{\le 2} \arrow[l, bend right, "\mathrm{Tr}_1"'] \arrow[r, hook] & \cdots \arrow[l, bend right] \arrow[r, hook] & \mathcal{E} \arrow[l, bend right]
\end{tikzcd}$$

Localization (or "reflection"):

$$\begin{tikzcd} \mathcal{D} \arrow[r, hook, "i"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathcal{E} \arrow[l, bend right, "\ell"'] \end{tikzcd}$$

can identify $\mathcal{D} \simeq i(\mathcal{D}) \subseteq \mathcal{E}$

Kernel of localization: $\mathcal{T} \subseteq \mathcal{E}^{\to}$

f in $\mathcal{T}$ iff $\ell(f)$ is iso

$\mathcal{T}$ is strongly saturated:

(1) Isos $\subseteq \mathcal{T}$

(2) "2-out-of-3"

$$\begin{tikzcd} & \cdot \arrow[dr, "g"] \arrow[d, phantom, "="] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "gf"'] & {} & \cdot \end{tikzcd}$$

any 2 in $\mathcal{T}$ $\Rightarrow$ all in $\mathcal{T}$

(3) Stable under colimits in $\mathcal{E}^{\to}$

$$\left\{ \begin{array}{c} \text{localizations} \\ \text{of } \mathcal{E} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} \text{strongly saturated} \\ \mathcal{T} \subseteq \mathcal{E}^{\to} \end{array} \right\}$$

$$(L, 1) \longmapsto \mathcal{T}$$

$$\mathcal{D} = \{ X \mid \operatorname{Map}(t, X) \text{ iso } \forall\, t \in \mathcal{T} \} \subseteq \mathcal{E}$$

If $\mathcal{E}$ presentable:

$$\left\{ \begin{array}{c} \text{accessible} \\ \text{localizations} \\ \text{of } \mathcal{E} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{l} \text{strongly saturated} \\ \mathcal{T} \subseteq \mathcal{E}^{\to} \text{ such that} \\ \mathcal{T} = \overline{S} \text{ for some set } S \end{array} \right\}$$

"strong saturation" of S

$$\left\{ \begin{array}{c} \text{accessible} \\ \text{left exact localizations} \\ \text{of } \mathcal{E} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{strongly saturated} \\ \mathcal{T} \subseteq \mathcal{E}^{\to} \text{ such that} \\ \mathcal{T} \text{ closed under base change} \\ \mathcal{T} = \overline{S},\ S \text{ set} \end{array} \right\}$$

Q: If $\mathcal{E}$ is an ∞-topos, what are the left exact localizations?

e.g.

Anel-Biedermann-Finster-Joyal, "Higher sheaves and lex-loc of ∞-topoi" (arXiv 2021)

$\mathcal{E}$ = ∞-topos, lex localization $\mathcal{D} \underset{\hookrightarrow}{\overset{\ell}{\underset{\perp}{\leftleftarrows}}} \mathcal{E}$, kernel $\mathcal{T} \subseteq \mathcal{E}^{\to}$

Fact: $\mathcal{T}_{loc} := \mathcal{T} \cap \mathrm{Cart}(\mathcal{E}^{\to})$ is a local class

$$\Longrightarrow \quad \begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ A \arrow[r, two heads, "p"'] & B \end{tikzcd} \quad \text{in } \mathcal{E}$$

p cover, $f \in \mathcal{T} \Longrightarrow g \in \mathcal{T}$

$\mathcal{E}$ = ∞-topos, <u>lex</u> localization $\mathcal{D} \underset{\perp}{\overset{\ell}{\leftleftarrows}} \mathcal{E}$, kernel $\mathcal{T}$:

Lemma:

$$\begin{tikzcd} & E \arrow[dr, "i"] & \\ A \arrow[ur, "p"] \arrow[rr, "f"'] & & B \end{tikzcd}$$

$f = ip$: $i \in \mathrm{Mono}$, $p \in \mathrm{Cov}$

$f \in \mathcal{T} \iff i, \Delta(f) \in \mathcal{T}$

Proof: $(\Rightarrow)$: easy. $\ell(i)$ mono, $\ell(f)$ iso $\Rightarrow i \in \mathcal{T}$, $\ell(\Delta(f)) = \Delta(\ell(f)) \Rightarrow \Delta(f) \in \mathcal{T}$

$(\Leftarrow)$: $\Delta(f)$: $A \xrightarrow{\Delta(p)} A \times_E A \overset{j}{\rightarrowtail} A \times_B A$. $i \in \mathcal{T} \Rightarrow j \in \mathcal{T} \Rightarrow \Delta(p) \in \mathcal{T}$.

$$\begin{tikzcd} A \arrow[r, "\Delta(p)"] \arrow[dr, "\mathrm{id}"'] & A \times_E A \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "p"] \\ & A \arrow[r, two heads, "p"', "\mathrm{cov}"] & E \end{tikzcd}$$

$\Delta(p) \in \mathcal{T} \Rightarrow q \in \mathcal{T} \underset{p \text{ cov}}{\Longrightarrow} p \in \mathcal{T} \underset{i \in \mathcal{T}}{\Longrightarrow} f \in \mathcal{T}$

Consequence: lex loc $\mathcal{O} \underset{i^*}{\overset{\ell}{\rightleftarrows}} \mathcal{E}$ = ∞-topos, kernel $\mathcal{T}$

$\forall n \leq \infty$ $\quad \mathcal{T} \cap \mathrm{Trunc}_n$ is determined by $\mathcal{T} \cap \mathrm{Mono}$

if $\Delta^{n+2}(f)$ iso: $\quad f \in \mathcal{T}$ iff $i_0, i_1, \dots, i_{n+1} \in \mathcal{T}$

$$\begin{tikzcd}
 & E_k \arrow[dr, hook, "i_k"] & \\
A \arrow[ur, twoheadrightarrow] \arrow[rr, "\Delta^k(f)"'] & & B
\end{tikzcd}$$

Topological localization of ∞-cat :

lex loc s.t kernel $\tau = \bar{S}$, $S \subseteq \mathrm{Mono}$
set

$\Rightarrow$ **Thm** : Every lex localization of an n-topos $(n<\infty)$ is topological

Proof : all maps in a n-category are (n-1)-truncated

Topological localizations of $PSh(C) = Fun(C^{op}, S)$:

Kernel $\mathcal{T} = \overline{S}$, S set $\subseteq$ Mono

- Mono local class $\Leftarrow$
- $\{p(c)\}$ representables are generators

$\Longrightarrow$ Can choose $S = \{ U \xrightarrow{i} p(c) \in \mathcal{T} \}$ "sieves"

$\Longrightarrow$ is a Grothendieck topology on C

($\Longleftrightarrow$ Grothendieck topologies on $h_1 C$)
$\uparrow$
C

Conclusion: Every topological localization of $PSh(C)$

$\underset{n=\infty}{=}$ is of form $Sh(C,S) \subseteq PSh(C)$

(C,S) an "∞-site" $\quad$ "$\{F \mid Map(s,F)$ iso $\forall\, s \in S\}$

$\underline{\underline{n<\infty}}$: Every n-topos is of the form

$$Sh(C,S)_{\leq n-1} \subseteq PSh(C)_{\leq n-1}$$

for some n-site (C,S)

n-category

Cotopological localization of presentable ∞-cat:

accesible lex localization, kernel τ such that

$$\tau \cap \mathrm{Mono} = \mathrm{Iso}$$

Prop: acc lex loc of ∞-topos is cotopological iff

$$\tau \subseteq \mathrm{Conn}_\infty$$

Proof of cotop $\Leftarrow$ $\mathcal{T} \subseteq \mathrm{Conn}_\infty$: $\mathrm{Conn}_\infty \cap \mathrm{Mono} = \mathrm{Iso}$

Proof of $\mathcal{T} \subseteq \mathrm{Conn}_\infty \Leftarrow$ cotop :   Show $\mathcal{T} \subseteq \mathrm{Conn}_n \ \forall n$

- $\mathcal{T} \subseteq \mathrm{Cover}$:

$$\begin{tikzcd} \cdot \arrow[r, "p"] \arrow[rr, bend right, "f"'] & \cdot \arrow[r, "i"] \arrow[d, phantom, "="] & \cdot \\ & {} & \end{tikzcd}$$

$f \in \mathcal{T}$, $\ell(i)$ mono, $\ell(f)$ iso

$\Rightarrow i \in \mathcal{T} \underset{\text{cotop}}{\Rightarrow} i$ iso, $\Rightarrow f \in \mathrm{Cov}$

- $\mathcal{T} \subseteq \mathrm{Conn}_n$, induction on n

<u>Use</u> : $f \in \mathrm{Cover}, n \geq 0$: $f \in \mathrm{Conn}_n \Leftrightarrow \Delta(f) \in \mathrm{Conn}_{n-1}$

Hypercompletion: $\mathcal{E}$ - ∞ topos

$X \in \mathcal{E}$ is **hypercomplete** if $\mathrm{Conn}_\infty \perp (X \to 1)$

Fact: $\underline{\mathcal{E}}^{hyp} \overset{\ell}{\hookrightarrow} \mathcal{E}$ is a cotopological localization

kernel $= \mathrm{Conn}_\infty$

Note: $(\mathcal{E}^{hyp})_{\leq n} = \mathcal{E}_{\leq n} \quad \forall \; n < \infty$

Example: $\underline{Sh(X)}$ of ~~previous lecture~~ earlier is not hypercomplete.

$\overset{\cup}{F}$ ∞-connected, $F \neq 1$

Thm (Lurie): Every acc lex localization of ∞-topos factors as

$$\begin{tikzcd} \mathcal{E}'' \arrow[r, hook] & \mathcal{E}' \arrow[l, bend right, "\ell''"'] \arrow[r, hook] & \mathcal{E} \arrow[l, bend right, "\ell'"'] \end{tikzcd}$$

cotopological topological

Cor: Every ∞-topos$^{\mathcal{E}}$ is a cotopological localization of some $Sh(C,S)$, (C,S) ∞-site

$$\begin{tikzcd} Sh(C,S) \arrow[d, shift right] \\ \mathcal{E} \arrow[u, shift right, "\text{cotop}"'] \end{tikzcd}$$

$\ni$ ℓ inverts some ∞-connected maps

Joyal, Jardine "simplicial presheaves" in 1-site (C,S)

$\simeq$ $Sh(C,S)^{hyp}$ "

Q: Is every ∞-topos equivalent to some $Sh(C,S)$? (C,S) ∞-site

"canonical site" of $\mathcal{E}$ might not have $\mathcal{E}$ as a topological loc.

Geometric morphism $\mathcal{E} \xrightarrow{f} \mathcal{F}$:

is adjoint pair
$$\begin{tikzcd} \mathcal{E} \arrow[r, shift right, "f_*"'] \arrow[r, phantom, "\perp"] & \mathcal{F} \arrow[l, shift right, "f^*"'] \end{tikzcd}$$
f^* lex

$\Rightarrow$ ∞-category of geometric morphisms

$$\mathrm{Fun}_*(\mathcal{E}, \mathcal{F}) = \mathrm{Fun}^*(\mathcal{F}, \mathcal{E})^{op}$$

$\Rightarrow$ ∞-category of ∞-topoi : $\infty\mathrm{Top}$ — large

$$\mathrm{Map}_{\infty\mathrm{Top}}(\mathcal{E}, \mathcal{F}) = \mathrm{Fun}_*(\mathcal{E}, \mathcal{F})$$

Computing geometric morphisms $\mathcal{E} \to PSh(C)$:

$$
\begin{tikzcd}
\mathrm{Fun}^*(PSh(C), \mathcal{E}) \arrow[r] & \mathrm{Fun}(PSh(C), \mathcal{E})^{\text{colim pres}} \arrow[d, "\simeq", "\mathrm{LKan}"'] \\
 & \mathrm{Fun}(C, \mathcal{E})
\end{tikzcd}
$$

$F : PSh(C) \to \mathcal{E}$ st.

(1) $F(1) = 1$

(2) F pres. pullbacks of

$$
\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \rho(c_2) \arrow[d] \\
\rho(c_1) \arrow[r] & \rho(c_0)
\end{tikzcd}
$$

Examples:

Terminal ∞-topos: $\exists!\ \mathcal{E} \xrightarrow{q} \underline{\underline{\mathcal{S}}}$

Point: $p : \mathcal{S} \longrightarrow \mathcal{E}$

$\mathcal{E}$ has enough points if $\{\underbrace{p^* : \mathcal{E} \to \mathcal{S}}_{\forall \text{ points}}\}$ jointly conservative
"stalks"

Warning: $\mathcal{S} \simeq \mathcal{S}^{hyp}$, $\mathrm{Conn}_\infty(\mathcal{S}) = \mathrm{Iso}(\mathcal{S})$

$\Rightarrow$ $\mathcal{E}$ can have "enough points" only if $\mathcal{E} \simeq \mathcal{E}^{hyp}$
(in this sense)

Slices:

$$\infty\mathrm{Top}(\mathcal{F}, \mathcal{E}_{/X}) \simeq \{ (f : \mathcal{F} \to \mathcal{E},\ (1 \to f^*X) \in \mathcal{F}) \}$$

$$\Rightarrow \quad \begin{aligned} \mathcal{E} &\hookrightarrow \infty\mathrm{Top}_{/\mathcal{E}} \\ X &\longmapsto \begin{pmatrix} \mathcal{E}_{/X} \\ \downarrow \\ \mathcal{E} \end{pmatrix} \end{aligned}$$

Torsors:

G - small ∞-groupoid (eg group)

$$\mathrm{Fun}^*(\mathrm{PSh}(G), \mathcal{E}) \xrightarrow{\simeq} \{P : G^{op} \to \mathcal{E} \mid \underset{G^{op}}{\mathrm{colim}}\, P \simeq 1\} \subseteq \mathrm{Fun}(G^{op}, \mathcal{E})$$

"G-torsor"

Why? (1) $\coprod_{x \in G_0} P(x) \longrightarrow 1$ cover in $\mathcal{E}$

$\pi : \mathcal{E} \to \mathcal{S}$

(2)

$$\begin{array}{ccc} P(x) \times \pi^* \mathrm{Map}_G(\underline{y}, x) & \longrightarrow & P(\underline{y}) \\ \downarrow & \lrcorner & \downarrow \\ P(x) & \longrightarrow & 1 \end{array}$$

$x, y \in G_0, \quad \pi : \mathcal{E} \to \mathcal{S}$

adm:

$$\begin{array}{ccc} \boxed{P \times_{\pi^*} G} & \longrightarrow & P \\ \text{proj} \downarrow & \lrcorner & \downarrow \\ P & \longrightarrow & 1 \end{array}$$

$\Big\uparrow$ descent

$\Big\downarrow$

$$P(x) \times \pi^* p(x) \longrightarrow \underline{P} \quad \Leftarrow \text{ Cart nat trans}$$

for each $x \in G_0$

$$G^{op} \longrightarrow \mathcal{E}$$

$\Downarrow \ \mathrm{colim}_{G^{op}}$

$$P(x) \longrightarrow 1$$

Nikolaus–Schreiber–Stevenson, "Principal bundles–general theory" (arXiv 2016) n-localic reflection:

∞-cat of ∞-topoi, n-topoi

$$\begin{tikzcd} \infty\mathrm{Top} \arrow[rr, bend left, "\mathcal{E} \mapsto \mathcal{E}_{\leq n-1}"] \arrow[rr, phantom, "\perp"] & & n\mathrm{Top} \arrow[ll, bend left, "r_n\mathcal{F} \mapsfrom \mathcal{F}"] \end{tikzcd}$$

Formula for n-localic reflection:

n-topos $\qquad\qquad\qquad\qquad\qquad\qquad$ ∞-topos

$$\mathcal{F} = \mathrm{Sh}(C,S)_{\leq n-1} \longmapsto r_n\mathcal{F} \simeq \mathrm{Sh}(C,S)$$

pick

(C,S) n-site

C has finite limits

Cohomology of an ∞-topos:

$$\mathcal{E} \xrightarrow{q} \mathcal{S} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{E} \arrow[r, shift right, "q_*"'] & \mathcal{S} \arrow[l, shift right, "q^*"'] \end{tikzcd}$$

q^* - constant

q_* - sections

$$K(G,n) \quad \rightsquigarrow \quad H^j(\mathcal{E}, G) := \pi_{n-j}\, q_* q^* K(G,n)$$

EM-space in $\mathcal{S}$

G = abelian group

$K(G,n) \in \mathcal{S}_{\leq n}$ → does not distinguish $\mathcal{E}$, $\mathcal{E}^{hyp}$

Shape: $\mathcal{E}$ ∞-topos, $\mathcal{E} \xrightarrow{q} \mathcal{S}$

$\leadsto$ $q^* \circ q_* :$ $\mathcal{S} \xrightarrow{q^*} \mathcal{E} \xrightarrow{q_*} \mathcal{S}$

object of $\mathrm{Pro}(\mathcal{S}) := \{ F : \mathcal{S} \to \mathcal{S} \mid \text{lex} \} \subseteq \mathrm{Fun}(\mathcal{S}, \mathcal{S})^{op}$

$$\uparrow \rho$$
$$\mathcal{S}$$

Thm: X paracompact topological space $\Longrightarrow$ $\mathrm{Shape}(\mathrm{Sh}(X)) = \rho(|X|)$

$|X| =$ ∞-groupoid = htpy type of X.

Toen, "Vers une interpretation Galoisienne..." (2002)
Lurie, HTT

Application: Sheaves of ∞-categories

$\mathcal{E}$: ∞-topos, $\mathcal{C}$: ∞-category, complete.

$\mathcal{E}$ 1-topos
$\Downarrow$
$n_1\mathcal{E}_{\infty\text{-topos}}$

$$\mathrm{Sh}_{\mathcal{C}}(\mathcal{E}) := \{ F: \mathcal{E}^{op} \to \mathcal{C}, \text{ pres all limits} \}$$

$\rightsquigarrow$ can define $\mathrm{Sh}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{E})$

Example: $X \supseteq \mathcal{U}$ scheme $\rightsquigarrow$ $U \mapsto \tilde{D}(U)$

$\tilde{D}(U)$ = ∞-cat$_n$

$\big\downarrow h_1$

$D(\mathcal{U})$ - derived cat of qCoh sheaves (not a sheaf)

Application: Derived geometry

$\mathcal{C}$:= an ∞-category of "ring-like objects"

e.g. comm dga, "$\mathbb{E}_\infty$-ring spectra"

$\rightsquigarrow$ Ringed ∞-topos: $(\mathcal{X}, \underline{\mathcal{O}})$

$\mathcal{X}$: ∞-topos
$\underline{\mathcal{O}}$ = sheaf on $\mathcal{X}$, values in $\underline{\underline{\mathcal{C}}}$

Toën, "Derived algebraic geometry" (2014)
Lurie, DAG I, II, III, etc. SAG

Application: Differential cohomology

$Sh(Man)$ – sheaves of ∞-groupoids on gros site of C^∞-manifolds

Contains: · $K(\mathbb{Z}, n)$ – represents singular cohomology

· Ω^n – represents differential forms

$\rightsquigarrow$ repr differential cohomology

Schreiber, "Differential cohomology in a cohesive ∞-topos

Application: Stratified ∞-topoi + constructible sheaves

Lurie, "Higher Algebra"

Barwick, Glasman, Haine, "Exodromy" (arXiv 2020)

Logical aspects:

"internal language of a 1-topos"

"Mitchell-Bénabou language" : a type theory

$\rightsquigarrow$ interpretation in an (elementary) 1-topos $\mathcal{E}$ ("Kripke-Joyal semantics")

types $\longleftrightarrow$ objects of $\mathcal{E}$

formulas $\longleftrightarrow$ morphisms $X \to \Omega$

dependent sums, products $\longleftrightarrow$ Σ_f, Π_f (adjoints to f^*)

Martin-Löf dependent type theory:

- identity type $x =_A y$ for any $x, y : A$
- type families $P : X \longrightarrow \mathrm{Type}$

$\rightsquigarrow$ "$x =_A y$" is like a "space of paths"

$\rightsquigarrow$ homotopy theoretic interpretation

- Awodey-Warren, "Homotopy theoretic models of identity types" (2009)
- Voevodsky, "Equivalence axiom + univalent models of type theory" (2010)

Univalence:

Two types of equivalence of functions $f, g : A \to B$ of types

- identity $f = g$
- homotopy equivalence $f \simeq g$
  ($h : g \circ f = id$, $k : f \circ g = id$)

Univalence axiom: these are the same

Voevodsky: "∞-groupoids are a model of U.T.T"

so that Type $\rightsquigarrow$ Ω (object classifier)

Kapulkin - Lumsdaine (Voevodsky)

"Every ∞-topos is a model for U.T.T"

Warning: Type theory has functions, composition, associative

model must be a 1-category.

e.g. a Q.M.C., whose core ∞-category is an ∞-topos

Shulman, "All $(\infty,1)$-toposes have strict univalent universes" (arXiv 2019)

Thank you!